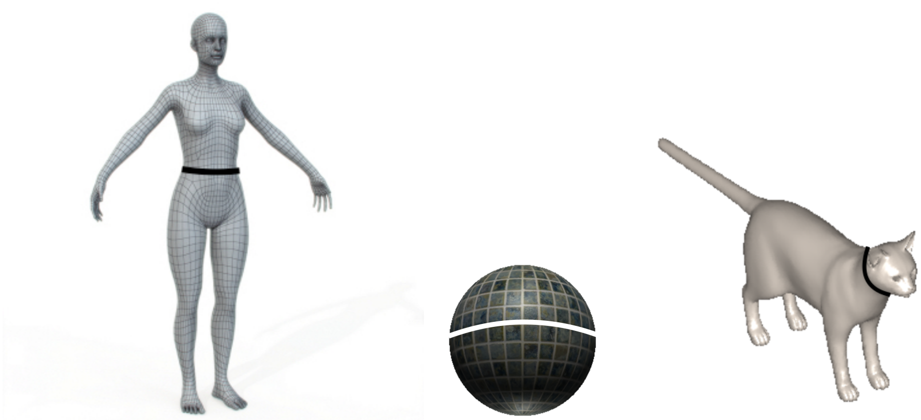
Main results of the project

Ideal fluid vortices, either regular or singular, are confined to their the coadjoint orbit of the group of volume-preserving diffeomorphisms.

Singular vortices of codimension 2 for 2D ideal fluid, point vortices, have been intensively studied over the years. In the articles [2,5] we dealt with 2D vorticity curves, i.e. codimension 1. We obtained the Onsager-Feynman prequantization condition of their coadjoint orbits: the product of the area bounded by the curve and the total vorticity must be an integer multiple of 2π. In addition we found two formulas for the associated character, one geometric and one algebraic. Combining the codimension 1 and 2 vortices, we get pointed vortex loops (examples of nonlinear drapes in small dimensions). In contrast, these coadjoint orbits always admit prequantizations.

A nonlinear flag in a differentiable manifold M consists of a collection of submanifolds embedded in each other: $N_1 \subset \cdots \subset N_k \subset M$, each subvariety being endowed with a volume form (or, more generally, with a density). In the paper [1] we studied the geometry of Fréchet differentiable manifolds of weighted nonlinear flags. We have described a class of coadjoint orbits of the group of Hamiltonian diffeomorphisms, coadjoint orbits formed from isotropic weighted nonlinear flags. The description includes a homology condition (related to the weights), but also uses Weinstein's isodrastic distribution.

Nonlinear flag manifolds also prove useful as shape spaces. In the paper [3] we study metrics of the elastic metric type on nonlinear flag manifolds consisting of $\mathbb{R}^3$ surfaces decorated with 1-dimensional (curved) drawings. Gauge invariant elastic metrics on the shape space of undecorated surfaces depends on the sum of the principal curvatures (mean curvature) and their difference. The proposed 6-parameter family of gauge invariant elastic metrics on the space of decorated surfaces with curves depend, in addition, on the geodesic and normal curvatures of the curve on the surface, as well as on its geodesic torsion.



Singular codimension 2 vortices for the ideal fluid (vorticity filaments) are well known. In the paper [4] we dealt with codimension 1: vortex sheets (N, β), where β is a Morse 1-form on the (hyper)surface $N \subset M$. Thus N is foliated into vortex filaments, except for a finite number of points. To describe their coadjoint orbits (in the decorated nonlinear Grassmannian), we have introduced a version of Weinstein's isodrastic distribution adapted to the volume form (instead of the symplectic form).

In general one obtains coadjoint orbits in the Ismagilov central extension of the group of (exact) volume preserving diffeomorphisms $\mathrm{Diff}_{\mathrm{vol}}(M)$. We found the necessary and sufficient condition to obtain coadjoint orbits relevant to the ideal fluid, i.e. in $\mathrm{Diff}_{\mathrm{vol}}(M)$ without central extension: de Rham's cohomology class $[\beta_N]$ annihilates all cohomology classes in $H^{n-2}(N)$ coming from M.

The Ismagilov extension integrates the Lichnerowicz central extension of the Lie algebra $\mathfrak{X}_{\rm vol}(M)$ of (exact) divergence free vector fields:

$$H^{n-2}(M) \longrightarrow \Omega^{n-2}(M)/\text{Im } d \longrightarrow \mathfrak{X}_{vol}(M).$$

Claude Roger's 1995 conjecture states that this extension is universal. In the preprint [6] we managed to prove this conjecture by taking a detour into the field of Leibniz algebras. More precisely, we have shown that

$$\text{Ker } d \longrightarrow \Omega^{n-2}(M) \longrightarrow \mathfrak{X}_{\rm vol}(M, \mu)$$

is a central extension of Leibniz algebras. The fact that the space of differential forms $\Omega^{n-2}(M)$, unlike its quotient space, is a space of sections in a vector bundle, allows us to apply Peetre's theorem.

Articole

[1] Stefan Haller, Cornelia Vizman - Weighted nonlinear flag manifolds as coadjoint orbits, Canadian Journal of Mathematics, 2023.
[2] Ioana Ciuclea, Cornelia Vizman - Pointed vortex loops in ideal 2D fluids, J. Phys. A: Math. Theor., 2023, 56 245201.
[3] Ioana Ciuclea, Alice Barbara Tumpach, Cornelia Vizman - Shape spaces of nonlinear flags, Geometric Science of Information: 6th International Conference, GSI 2023, St.Malo, France, 30.08-1.09.2023, Proceedings, Part I, 41-50 (the prize winning paper of GSI 2023).
[4] Francois Gay-Balmaz, Cornelia Vizman - Coadjoint orbits of vortex sheets in ideal fluids, preprint.
[5] Francois Gay-Balmaz, Cornelia Vizman - Vortex loops and characters in 2D fluids, preprint.
[6] Bas Janssens, Leonid Ryvkin, Cornelia Vizman, Universal central extension of the Lie algebra of exact divergence-free vector fields, preprint.